MR1328251 (96g:32037) 32G20 (14J32 14N10 32G81 32J18 33C70)
Batyrev, Victor V. (D-ESSN); **van Straten, Duco** (D-KSRL)
Generalized hypergeometric functions and rational curves on Calabi-Yau complete intersections in toric varieties. (English summary)
Comm. Math. Phys. **168** (1995), *no. 3,* 493–533.

One of the most important implications of mirror symmetry is its prediction of the numbers of rational curves on Calabi-Yau manifolds. Originally [cf. P. Candelas et al., Nuclear Phys. B **359** (1991), no. 1, 21–74; MR1115626 (93b:32029)] the process of obtaining these numbers was carried out via constructing another (mirror) manifold, finding differential equations for the periods of holomorphic forms on the latter and taking certain expansions of the quantities associated with these forms (Yukawa couplings) relative to parameters constructed using differential equations. This process was carried out (with some steps only conjecturally) in many cases, though mostly for Calabi-Yau manifolds which are hypersurfaces in weighted projective spaces. In the note by the reviewer and J. Teitelbaum [Internat. Math. Res. Notices **1993**, no. 1, 29–39; MR1201748 (93k:14013)] predictions for the number of rational curves on 3-dimensional complete intersections in ordinary projective space were made by guessing the differential equation for the periods of the holomorphic form on the mirror without actually finding corresponding mirror manifolds.

In the present paper the authors substantially expand this in several directions. They conjecture what should be the power series $\Phi_0(z)$ for the holomorphic solution of the differential equation for the period of a holomorphic form on the mirror in terms of the original Calabi-Yau manifold in the case when the Calabi-Yau is a complete intersection in a toric variety. For example, for a complete intersection of hypersurfaces of degree $d_1, \cdots, d_r$ in $\mathbf{C}\mathrm{P}^{n+r}$ the corresponding series is the generalized hypergeometric series

$$\sum_{i=0}^{i=\infty} \frac{(nd_1)!}{(n!)^{d_1}} \cdots \frac{(nd_r)!}{(n!)^{d_r}} z^n.$$

In the general case of toric varieties one has a more involved expression depending on the fan defining the toric variety. The authors also provide a beautiful alternative description of the series $\Phi_0(z)$ in terms of intersection numbers of nef curves on the toric variety with the hypersurfaces, defining the Calabi-Yau complete intersection in question.

This power series $\Phi_0(z)$ is central for the predictions of the rational curves since it yields in many examples the differential equations for the periods of forms on mirrors, the Yukawa couplings and the parameters needed for the expansion, the coefficients of which are the numbers of rational curves. Following their goal to recast essential features of the calculations yielding the predictions, the authors conjecture that Φ_0 is a solution to the differential system defined by the Witten connection in quantum cohomology (which should be the case if there is identification of superconformal field theories since the latter should yield an identification of the Gauss-Manin and Witten connections on their respective domains). They give a conjectural expression for q-

coordinates needed for the expansion of the Yukawa couplings and point out that q-series should have integral coefficients in terms of original coordinates. They also remark that their generalized hypergeometric series belong to a subclass of the generalized hypergeometric functions with the resonance considered by Gel′fand-Kapranov-Zelevinsky.

The series $\Phi_0(z)$ has an important interpretation as an integral:

$$\frac{1}{(2\pi i)^{n+r}}\int_{|X_j|=1}\frac{1}{\prod_{l=1}^{k}P_l(X_1,\cdots,X_{d+r})}\frac{dX_1}{X_1}\cdots\frac{dX_{i+r}}{X_{i+r}},$$

where $P_i(X_1,\cdots,X_{i+r})$ are Laurent polynomials determined by the combinatorial data defining the complete intersection and depending on parameters which also provide dependence of Φ on z. The P_i's conjecturally define the mirror. The authors conjecture that the mirrors of complete intersections are the Calabi-Yau compactifications of the complete intersections given by the equations $P_1(X_1,\cdots,X_{d+r})=\cdots=P_r(X_1,\cdots,X_{d+r})=0$ in a torus $\mathbf{T}$ with coordinates $(X_1,\cdots,X_{d+r})$.

In the final part of the paper the authors give ample evidence for these conjectures. They provide numerous calculations of their generalized hypergeometric series Φ_0 for complete intersections in products of projective spaces and del Pezzo surfaces. Then they calculate coefficients of expansions of Yukawa couplings and compare these coefficients with ad hoc calculations of the number of lines and quadrics confirming the predictions.

Since this paper was written several new developments have taken place. L. Borisov ["Towards the mirror symmetry for Calabi-Yau complete intersections in Gorenstein toric Fano varieties", Preprint, http://xxx.lanl.gov/abs/alg-geom/9310001] came up with the construction of mirrors of complete intersections of toric varieties, S. Hosono et al. [Comm. Math. Phys. **167** (1995), no. 2, 301–350; MR1316509 (96a:32044); Nuclear Phys. B **433** (1995), no. 3, 501–552; MR1319280 (96d:32028)] and P. Candelas et al. [Nuclear Phys. B **416** (1994), no. 2, 481–538; MR1274436 (95k:32020)] worked out the differential systems for the periods of holomorphic forms, and progress was made on the integrality conjectured here of the q-expansions mentioned above by B. H. Lian and S.-T. Yau [Comm. Math. Phys. **176** (1996), no. 1, 163–191].

Reviewed by *Anatoly Libgober*